DEFECTIVENESS AND IDENTIFIABILITY: A GEOMETRIC POINT OF VIEW ON TENSOR ANALYSIS

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Ph.D. Thesis Abstract

1 Introduction

Given a collection of vector spaces $V_1, \dots, V_k$ we can construct their tensor product, namely

$$V_1 \otimes \cdots \otimes V_k$$

An element $T \in V_1 \otimes \cdots \otimes V_k$ is said to be a tensor. In a schematic way we can think of a tensor as a multidimensional array, generalizing the concept of ordinary vectors v and rectangular matrices $A_{i,j}$. Due to their natural structure, tensors are useful for collecting numerical data and organizing them in a clever way. For this reason tensors are ubiquitous in both applied and pure algebraic geoemetry.

Here we briefly sketch some relevant examples of their range of applications:

1 **Quantum Computing** A quantum system of q-bits is governed by the entangle of N quantum bit states $|\psi_1\rangle, \dots, |\psi_N\rangle$, where each $|\psi_i\rangle \in \mathbb{C}^2$ is a complex vector of norm 1. In particular the entire ensemble is parametrized by tensors of the form

$$|\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle \in (\mathbb{C}^2)^{\otimes N}$$

The study of decompositions of these particular tensors is crucial in order to understand the quantum behaviour of the system.

2 **Statitstics and Gaussian distributions** Given a Gaussian mixture model $\mathcal{G}$, i.e. a sum of Gaussian normal distributions $Y_i \sim \mathcal{N}(l, q)$, we hope to infer the various degree moments of the single normal distributions Y_i knowing the specific moments of the Gaussian mixture model $\mathcal{G}$. This problem shows to be equivalent to controling the decomposition of specific symmetric tensors.

3 **Algorithmic Problems** The complexity of a tensor governs the computational cost of several algorithms in linear algebra. The major example is the matrix multiplication tensor: being able to know the terms of a minimal decomposition is equivalent to compute the classical matrix multiplication as fast as possible. This is essential for example in the field of Signal Processing.

We will always work over the field of complex numbers $\mathbb{C}$.

From the examples above is clear that the core problem of tensor geometry is the control of the decompositions of a given tensor. Let us set the problem in a more geometric terms.

Given a tensor $T \in V_1 \otimes \cdots \otimes V_k$ we define a decomposition as

$$T = \sum_{i=1}^{r} \lambda_i T_i \tag{1.0}$$

where each $T_i = v_1^i \otimes \cdots \otimes v_k^i$ is a tensor of rank 1.

Since the rescaling factors λ_i in equation 1.0 do not change the structure of the decomposition we can restrict ourselves to the projective case.

In particular if $X \subset \mathbb{P}^N$ is a projective variety parametrizing special type of tensors (*tensor varieties*), where $\mathbb{P}^N = \mathbb{P}(\Gamma)$ for some submodule $\Gamma \subseteq V_1 \otimes \cdots \otimes V_s$, a crucial problem is to determine the dimension of the locus of tensors $T \in \mathbb{P}^N$ such that T can be expressed as a limit of linear combinations of "simpler" tensors, i.e. the ones belonging to the variety X. (looking at equation 1.0 it is easy to recognize X as the variety of rank 1 tensors.

More generally we can consider the $h-$secant variety $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$ attached to a projective variety $X \subset \mathbb{P}^N$, i.e. the Zariski closure of the points $z \in \mathbb{P}^N$ lying in the linear span of h points of X:

$$\mathbb{S}ec_h(X) = \overline{\{z \in \mathbb{P}^N | z \in \langle x_1, \dots, x_h \rangle \; s.t. \; x_1, \dots, x_h \in X\}} \subset \mathbb{P}^N$$

By a straightforward dimensional count the expected dimension of $\mathbb{S}ec_h(X)$ is

$$\operatorname{expdim}(\mathbb{S}ec_h(X)) = \min\{h \dim(X) + h - 1, N\}$$

The actual dimension of $\mathbb{S}ec_h(X)$ however can be smaller than the expected one, i.e.

$$\dim(\mathbb{S}ec_h(X)) < \operatorname{expdim}(\mathbb{S}ec_h(X))$$

In this case we say that X is $h-$defective.

The classification of $h-$defective tensor varieties is far from being complete. The only class completely classified is that of Veronese varieties. For other types of tensor varieties bounds on h such that X is not $h-$defective are given, see for instance and many others.

It is possible to control the $h-$secant defect studying the $h-$tangential projection

$$\tau_h : X \dashrightarrow \mathbb{P}^M$$

where τ_h is the rational map associated to the linear system $|\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2, \dots, x_h^2}|$. In particular if τ_h is generically finite then X is not $(h+1)-$defective.

Rephrasing the idea of equation 1.0, if $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ is a tensor variety a central problem to address is whenever a given tensor $T \in \mathbb{S}ec_h(X)$ can be written uniquely as

$$T = \lambda_1 T_1 + \cdots + \lambda_h T_h$$

where $T_1, \dots, T_h$ are tensors in X. This leads to the notion of identifiability for a tensor variety, and more generally for every variety $X \subset \mathbb{P}^N$. An element p of a projective space $\mathbb{P}^N$ is h-identifiable via a variety X if there is a unique way to write p as linear combination of h elements of X. In the classical setting this very often translates into rationality problems and it is linked to the existence of birational parameterizations.

In order to study identifiability properly we need to introduce notion of $h-$tangential weak defectiveness. Given a projective variety $X \subset \mathbb{P}^N$ and $x_1, \dots, x_h$ general points of X we denote by Γ_h the union of the irreducible components of the singular locus of the scheme $\langle \mathbb{T}_{x_1} X, \dots, \mathbb{T}_{x_h} X \rangle \cap X$ passing through the points $x_1, \dots, x_h$. The variety X is said to be $h-$tangential weakly defective is $\dim(\Gamma_h) > 0$, and Γ_h is called the $h-$tangential contact locus. If X is not h-tangential weakly defective then it is identifiable . Even though the converse does not hold in general, the h-tangential contact locus of X gives the right information on the number of decompositions of the general element of $\mathbb{S}ec_h(X)$, in fact the h-secant degree of X is equal to the h-secant degree of the h-tangential contact locus of X.

A common problem in tensor analysis is to compute the rank of a tensor whenever it is known a decomposition of it lying in a suitable tensor subvariety. In particular if $T \in \langle Y \rangle$ and h is the minimum such that

$$T = \lambda_1 T_1 + \cdots + \lambda_h T_h$$

with $Y \subset X$ a tensor subvariety and $T_i \in Y$, it could exist tensors $T'_1, \ldots, T'_s$ in X such that

$$T = \beta_1 T'_1 + \cdots + \beta_s T'_s$$

and $s < h$.

Comon's conjecture states that $h = s$ in the case where Y is a Veronese variety and X is a Segre variety.

More precisely, Comon's conjecture predicts that the rank of a homogeneous polynomial $F \in k[x_0, \ldots, x_n]_d$ with respect to the Veronese variety V_d^n is equal to its rank with respect to the Segre variety $SV_1^{\boldsymbol{n}} \cong (\mathbb{P}^n)^d$ into which V_d^n is diagonally embedded, that is

$$\operatorname{rank}_{V_d^n}(F) = \operatorname{rank}_{SV_1^{\boldsymbol{n}}}(F)$$

2 Results

Comon's Conjecture

In the first part, based on the paper [CMM] published in Mathematics (2018), we introduce a method to improve a classical result on Comon's conjecture. By standard arguments involving catalecticant matrices it is not hard to prove that Comon's conjecture holds for the general polynomial in $k[x_0, \ldots, x_n]_d$ of symmetric rank h as soon as $h < \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. We manage to improve this bound looking for equations for the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(V_d^n)$, not coming from catalecticant matrices, that are restrictions to the space of symmetric tensors of equations of the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(SV_1^{\boldsymbol{n}})$. We will do so by embedding the space of degree d polynomials into the space of degree $d+1$ polynomials by mapping F to $x_0 F$ and then considering suitable catalecticant matrices of $x_0 F$ rather than those of F itself.

Implementing this method in Macaulay2 we are able to prove for instance that Comon's conjecture holds for the general cubic polynomial in $n+1$ variables of rank $h = n+1$ as long as $n \leqslant 30$. Note that for cubics the usual flattenings work for $h \leqslant n$.

The main result of this Chapter is the following:

Theorem. Assume $n \geqslant 2$ and set $h = \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. Then Comon's conjecture holds for the general degree d homogeneous polynomial in $n+1$ variables of rank h in the following cases:

- $d = 3$ and $2 \leqslant n \leqslant 30$;
- $d = 5$ and $3 \leqslant n \leqslant 8$;
- $d = 7$ and $n = 4$.

Identifiability for subgeneric ranks

In the second part, based on the paper [CM1] published in JEMS in 2022, we develop an entirely new approach to study generic identifiability, based on a carefult study of the tangential contact locus. We derive identifiability statements for non secant defective varieties. Even if new this is not really surprising since weak defectiveness and tangential weak defectiveness, thanks to the Terracini Lemma, have secant defectiveness as a common ancestor. With this new approach we are able to translate all the literature on defective varieties into identifiability statements, providing in many cases sharp classification of h-identifiability.

The main result is the following theorem:

Theorem. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety of dimension n. Assume that:

a) X is k-twd,

b) X is not $(k-1)$-twd

c) $k > n$ and $N \geqslant (k+1)(n+1)-1$.

Then π^X_{k+1} is of fiber type.

Thanks to this theorem we are able to classify completely the subgeneric identifiability for Segre varieties that are product of $\mathbb{P}^1$'s:

Theorem. The Segre embedding of n copies of $\mathbb{P}^1$, with $n \geqslant 5$ is h-identifiable for any $h \leqslant \lfloor \frac{2^n}{n+1} \rfloor - 1$.

Recall that the generic rank of the Segre embedding of $(\mathbb{P}^1)^n$ is $\lceil \frac{2^n}{n+1} \rceil$, therefore our result shows generic identifiability of all sub-generic binary tensors, qbits in the quantum computing dictionary, in the perfect case, that is when $\frac{2^n}{n+1}$ is an integer, and all but the last one in general, as predicted by the conjecture posed by Chiantini.

Identifiability for the generic rank

In the third part, based on the paper [CM2] published in IMRN in 2021, we move to the study of generic identifiability. In particular generic identifiability of symmetric tensors has shown its close connection to modern birational projective geometry and especially to the maximal singularities methods.

We extend this theory to arbitrary tensors. As for the symmetric case it is expected that identifiability is very rare and our result support this convincement.

The main technical result of this part is the following:

Theorem. Let $X \subseteq \mathbb{P}^N$ be a projective irreducible, reduced and non-degenerate variety of general rank g. Let $\{x_1, ..., x_{g-1}\}$ be general points on X and $\mathcal{H} = \mathcal{H}(g-1)$. Assume that:

- X is perfect and non defective
- X is not $(g-1)$-twd

Then there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}$ such that for any point $y \in Y$

$$\operatorname{mult}_y D < 1 + \epsilon.$$

Thanks to the implementation of the so called Noether-Fano inequalities we were able to prove the non generic identifiability for many partially symmetric tensors:

Theorem. Fix two multiindexes $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$. Let $X = SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ the corresponding Segre-Veronese variety. Assume that $d_i > n_i + 1$, for $i = 1, \dots, r$, and

$$\lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i).$$

Then X is not generically identifiable.

Identifiability for Flag varieties

In the fourth part, based on the papers [CFM1] published in Journal of Pure and Applied Algebra in 2022 and [CFM2] published in Annali della Scuola Normale Superiore di Pisa, we investigate secant defectiveness of flag varieties applying the machinery of osculating projections.

Furthermore, our results on secant defectiveness, combined with a recent results of the second part, allow us to produce a bound for identifiability of flag varieties. Our main result can be summarized as follows:

Theorem. Consider a flag variety $\mathbb{F}(k_1, \dots, k_r; n)$. Assume that $n \geqslant 2k_j + 1$ for some index j and let l be the maximum among these j. Then, for

$$h \leqslant \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1) \rfloor},$$

$\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective. Furthermore, if for such h we have

$$h > 2\dim(\mathbb{F}(k_1, \dots, k_r; n)$$

then the general point of the h-secant variety of $\mathbb{F}(k_1, \dots, k_r; n)$ is h-identifiable.

Here is a list of the papers published extracted from the thesis.

References

[CFM1] Casarotti, A.; Freire, Barbosa A.; Massarenti, A.; *On secant dimensions and identifiability of Flag varieties* Journal of Pure and Applied Algebra, 2022, 226(6), 106969

[CFM2] Casarotti, A.; Freire, Barbosa A.; Massarenti, A.; *On tangential weak defectiveness and identifiability of projective varieties* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 2021, 22(4), pp. 1621–1642

[CM1] Casarotti, A.; Mella, M.; *From non defectivity to identifiability* Journal of the European Mathematical Society, 2022, DOI 10.4171/JEMS/1198

[CM2] Casarotti, A.; Mella, M.; *Tangential weak defectiveness and generic identifiability* International Mathematics Research Notices, rnab091, 2021, https://doi.org/10.1093/imrn/rnab091

[CMM] Casarotti,A.; Massarenti, A.; Mella, M.; *"On Comon's and Strassen's conjectures"* Mathematics, 2018, 6(11), 217