Summary

This thesis is divided in two parts. In the first part we consider spin systems in the d-dimensional lattice $\mathbb{Z}^d$ satisfying the so-called strong spatial mixing condition (SSM), a standard condition corresponding to exponential decay of correlations with distance between spins on the lattice. We show that the relative entropy functional of the corresponding Gibbs measure satisfies a family of inequalities which control the entropy on a given region $V \subset \mathbb{Z}^d$ in terms of a weighted sum of the entropies on blocks $A \subset V$ when each A is given an arbitrary nonnegative weight α_A. These inequalities generalize the well known logarithmic Sobolev inequality for the Glauber dynamics. Moreover, they provide a natural extension of the classical Shearer inequality satisfied by the Shannon entropy. Finally, they imply a family of modified logarithmic Sobolev inequalities which give tight bounds on the mixing time of arbitrary weighted block dynamics of heat bath type. Then, we show that the tools involved in the analysis of the block dynamics can be adapted to the study of the mixing time of the Swendsen-Wang dynamics for the ferromagnetic Ising and Potts models on the integer lattice $\mathbb{Z}^d$. This dynamics is a widely used Markov chain that has largely resisted sharp analysis because it is non-local, i.e., it changes the entire configuration in one step. In particular we prove that, whenever SSM holds, the mixing time on any n-vertex cube in $\mathbb{Z}^d$ is $O(\log n)$, and we prove this is tight by establishing a matching lower bound on the mixing time. The previous best known bound was $O(n)$. The proof of this fact utilizes a new factorization of the entropy in the joint probability space over spins and edges that underlies the Swendsen-Wang dynamics, which extends to general bipartite graphs of bounded degree. Our result then follows from the fact that this factorization implies the modified log-Sobolev inequality. This factorization also leads to several additional results, including mixing time bounds for a number of natural local and non-local Markov chains on the joint space, as well as for the standard random-cluster dynamics.

We finally extend our analysis to spin systems on an arbitrary graph $G = (V, E)$ with finite spin space, in which case we prove that a contractive coupling for an arbitrary local Markov chain implies optimal bounds on the mixing time and the modified log-Sobolev constant for a large class of Markov chains including the Glauber dynamics, arbitrary heat-bath block dynamics, and the Swendsen-Wang dynamics. This reveals a novel connection between probabilistic techniques for bounding the convergence to stationarity and analytic tools for analyzing the decay of relative entropy. As a corollary of our general results, we obtain $O(n \log n)$ mixing time and $\Omega(1/n)$ modified log-Sobolev constant of the Glauber dynamics for sampling random q-colorings of an n-vertex graph with constant maximum degree Δ when $q > (11/6 - \epsilon_0)\Delta$ for some fixed $\epsilon_0 > 0$. We also obtain $O(\log n)$ mixing time and $\Omega(1)$ modified log-Sobolev constant of the Swendsen-Wang dynamics for the ferromagnetic Ising model on an n-vertex graph of constant maximum degree when the parameters of the system lie in the tree uniqueness region. At the heart of our results are new techniques for establishing spectral independence of the spin system and block factorization of the relative entropy. Roughly speaking, a distribution is spectrally independent if the maximum eigenvalues of the influence matrices associated to the distribution and its conditional distributions are upper bounded. On one hand we prove that

a contractive coupling of any local Markov chain implies spectral independence of the Gibbs distribution. On the other hand we show that spectral independence implies factorization of entropy for arbitrary blocks, establishing optimal bounds on the modified log-Sobolev constant of the corresponding block dynamics.

The second part is devoted to the study of a nonlinear recombination model from population genetics as a combinatorial version of the Kac-Boltzmann equation from kinetic theory. Following Kac's approach, the nonlinear model is approximated by a mean field linear evolution with a large number of particles. In our setting, the latter takes the form of a generalized random transposition dynamics. Our main results establish a uniform in time propagation of chaos with quantitative bounds, and a tight entropy production estimate for the generalized random transpositions, which holds uniformly in the number of particles. As a byproduct of our analysis we obtain sharp estimates on the speed of convergence to stationarity for the nonlinear equation, both in terms of relative entropy and total variation norm.

The first part of this thesis is based on [3, 2, 1], while the second part is based on [4].

References

[1] Antonio Blanca, Pietro Caputo, Zongchen Chen, Daniel Parisi, Daniel Štefankovič, and Eric Vigoda. On mixing of Markov chains: coupling, spectral independence, and entropy factorization. *Electronic Journal of Probability*, 27(none):1 – 42, 2022.

[2] Antonio Blanca, Pietro Caputo, Daniel Parisi, Alistair Sinclair, and Eric Vigoda. Entropy decay in the Swendsen–Wang dynamics on $\mathbb{Z}^d$. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1551–1564, 2021.

[3] Pietro Caputo and Daniel Parisi. Block factorization of the relative entropy via spatial mixing. *Communications in Mathematical Physics*, 388:793–818, 2021.

[4] Pietro Caputo and Daniel Parisi. Nonlinear recombinations and generalized random transpositions. *arXiv preprint arXiv:2207.04775*, 2022.